Convective Instability in Porous Media with Throughflow

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In their study of the linear stability limits for free convection in layers of porous media or packed beds with throughflow, Jones and Persichetti (1986) found that throughflow was generally stabilizing, but there was an exception. In the case of flow toward a porous boundary, they found that the numerically calculated Rayleigh-Darcy number initially decreased and went through a minimum as the appropriate Peclet number increased from zero. Jones and Persichetti stated that the origin of the minimum was not completely understood, but they had eliminated numerical inaccuracy as a possible cause. Thus a small amount of throughflow can have a destabilizing effect in at least one situation. The primary purpose of this note is to explain this effect. The opportunity is taken to provide some supplementary material.

Analysis of Effect of Throughflow

In this paper the notation of Jones and Persichetti is followed, except that here their modified thermal Peclet number Pe will be denoted by P, and their parameter λ (which within the Boussinesq approximation is identical with their Rayleigh-Darcy number Ra) will be denoted by R. It is the critical value of the latter, R_c, which is of interest. When P is large, R_c increases with P for any specified set of boundary conditions. Then the effect of throughflow is to confine significant thermal gradients to a thermal boundary layer at the boundary toward which the throughflow is directed. The effective length scale, L, is then the small boundary layer thickness, rather than the layer thickness, and so the effective Rayleigh number, which is proportional to L, is much less than the actual Rayleigh number, R. Larger values of R are thus needed before convection begins. The effect of throughflow is then stabilizing. Within the bulk of the medium a large part of the heat transport can be effected by the throughflow alone, and the value at which the temperature gradient becomes so large that convection cells are required to transport heat is increased. Also, the effective Rayleigh number is then largely independent of the boundary conditions at the other boundary, from which the throughflow occurs. This was noted by Jones and Persichetti, and Homsy and Sherwood, 1976. Thus the situation pertaining to large values of P is well understood.

When P is small, the situation is not immediately clear and detailed analysis is needed. Following Jones and Persichetti, we can write the governing differential equations for the mass flux perturbation Γ and the temperature perturbation θ as

$$(D^2 - a^2)\,\Gamma = -Ra^2\,\theta \tag{1}$$

$$[PD - (D^2 - a^2)]\theta = -F(z)\Gamma \tag{2}$$

Here $D \equiv d/dz$, and z is the dimensionless vertical coordinate. The porous layer is bounded below at $z = 0$ and above at $z = 1$. The parameter a is the dimensionless horizontal wave number of the disturbance, and $F(z)$ is the dimensionless basic temperature gradient, which for the steady state (conduction and throughflow only) is given by

$$F(z) = \frac{-Pe^{Pz}}{e^P - 1} \tag{3}$$

Here $F(z)$ is defined so that it is negative for a layer heated from below. Positive values of P correspond to throughflow in the upward direction. At each boundary one condition on Γ and one condition on θ must be specified. There are various cases of interest. At an "impermeable" boundary (constant mass flux, and so impermeable to perturbations), $\Gamma = 0$. At a "porous" boundary (constant pressure), $D\Gamma = 0$. At a "conducting" boundary (constant temperature, and so zero perturbation temperature), $\theta = 0$. At an "insulating" boundary (constant heat flux, and so zero perturbation heat flux), $D\theta = 0$. In the previous studies of throughflow effects both boundaries have been taken to be conducting. In this paper both boundaries are taken to be insulating. These are interesting in their own right (as a limiting case approximated in some practical situations), but they are vital here because they allow a solution of the eigenvalue problem in simple closed form. This arises because the minimum value of R, as the wave number a varies, occurs at $a = 0$. It turns out that this means that the usual Galerkin method, which usually yields only an approximate value of R, now yields an exact value. In this respect the situation is analogous to that discussed in section 5 of Nield (1975).

For brevity, only the Galerkin analysis is given here, and that in sketch form only. A detailed exposition of the method may be found in Finlayson (1972). One puts $\Gamma = A\Gamma_1, \theta = B\theta_1$, where Γ_1 and θ_1 are trial functions that satisfy the boundary conditions, and substitutes into Eqs. 1 and 2. One then multiplies Eq. 1 by Γ_1, Eq. 2 by θ_1, and integrates each term from $z = 0$ to $z = 1$ and performs some integration by parts. The elimination of the arbitrary constants A and B then yields an equation for R, which may be written, when subscripts are dropped,

$$R = \frac{-\langle[(D\Gamma)^2 + a^2\Gamma^2]\rangle \{P\langle\theta D\theta\rangle + \langle[(D\theta)^2 + a^2\theta^2]\rangle\}}{a^2\langle\theta\Gamma\rangle\langle\theta F\Gamma\rangle} \quad (4)$$

where $\langle f \rangle$ denotes $\int_0^1 f\,dz$. The different sets of boundary conditions must now be considered.

Case 1: Both boundaries impermeable and insulating

$$\Gamma = 0, D\theta = 0 \text{ at } z = 0, 1.$$

The appropriate trial functions are $\Gamma = z - z^2, \theta = 1$, and Eq. 4 reduces to:

$$R = \frac{2P^2(1 + a^2/10)}{P\coth(P/2) - 2}$$

Clearly, R has a minimum when $a = 0$, and the minimum value is

$$R_c = \frac{2P^2}{P\coth(P/2) - 2}$$

Thus R_c is an even function of P, and for positive P is an increasing function of P. Hence throughflow is stabilizing for all P, and the direction of this flow does not matter. As $|P| \to \infty$, $R_c \sim 2|P|$. For small values of P, $R_c \sim 12 + P^2/5$.

Case 2: Both boundaries porous and insulating

$$D\Gamma = 0, D\theta = 0 \text{ at } z = 0, 1.$$

The appropriate trial functions are $\Gamma = 1, \theta = 1$, and now $R = a^2$ for all values of P. Hence $R_c = 0$.

Convection begins at an arbitrarily small temperature difference across the layer whether or not there is throughflow.

Case 3: Lower boundary impermeable and insulating, upper boundary porous and insulating

$$\Gamma = 0, D\theta = 0 \text{ at } z = 0,$$

$$D\Gamma = 0, D\theta = 0 \text{ at } z = 1$$

The appropriate trial functions are $\Gamma = z - z^2/2, \theta = 1$, and Eq. 4 gives

$$R_c = \frac{2P^2(e^P - 1)}{2P + 2 + e^P(P^2 - 2)}$$

For small values of P, $R_c \sim 3(1 - P/8)$. As P increases from zero, R_c decreases monotonically from the value 3 to the asymptotic value 2. Thus upflow is destabilizing for this case. As P decreases from zero, R_c increases monotonically, and $R_c \sim |P|$ as $P \to -\infty$. (This asymptotic expression is one-half the corresponding expression in case 1). Hence downflow is stabilizing. As expected, the value of R_c at any value of P for case 3 is intermediate between the corresponding values for cases 1 and 2.

Case 4: Lower boundary porous and insulating, upper boundary impermeable and insulating

Since the transformation $z \to 1 - z$, $P \to -P$ leaves the eigenvalue system of equations invariant, the results for this case can be deduced from those of case 3 very simply. One merely has to change the sign of P in the expression for R_c, which is equivalent to exchanging "upflow" and "downflow."

Discussion

Although the present results, for insulating boundaries, have some features distinct from those for conducting boundaries obtained by Jones and Persichetti, the overall picture is similar. For the symmetrical situations, where the lower and upper boundaries are of the same type, the critical Rayleigh number is an even function of P and throughflow is stabilizing by a degree that is independent of flow direction. When the two boundaries are of different types, throughflow in one direction will be destabilizing, for at least small values of P, since dR_c/dP at $P = 0$ is not zero.

The destabilization occurs when the throughflow is away from the more restrictive boundary (the impermeable boundary here). The throughflow then decreases the temperature gradient near the restrictive boundary and increases it in the remainder of the medium. Effectively, the applied temperature difference acts across a layer of smaller thickness, but the stabilizing effect of this change is more than made up by the destabilization produced by changing the effective boundary condition to a less restrictive one. Thus the destabilization produced by a small amount of throughflow arises because the linear temperature profile in the steady state is replaced by a nonlinear profile. The present problem may be regarded as a special case of the problem discussed by Nield (1975).

An essentially equivalent explanation for the destabilization is that the linear temperature profile is distorted by the throughflow so that the largest values of the basic temperature gradient occur in that part of the medium where the vertical perturbation velocity takes its largest values, and this leads to an increase in the rate at which energy is supplied to a disturbance. A measure of this is given by the expression $\langle\theta F\Gamma\rangle$ that appears in Eq. 4.

In this paper the eigenvalue of R has been obtained using a heuristic Galerkin technique. The same final expression for R can be obtained in a more formal manner by means of a series expansion in the powers of a^2. The expression $\langle\theta F\Gamma\rangle$ then arises when one imposes the condition (a solvability requirement) that the first-order solution be orthogonal to the zero-order solution.

The case considered in this paper, namely that where both boundaries are insulating, serves very well to discuss the effect of curvature of the steady-state temperature profile via the term $\langle \theta F \Gamma \rangle$, which appears as a multiplicative factor in Eq. 4. In general the situation is more complicated because the term $P \langle \theta D \theta \rangle$, which appears as an addition to a nonnegative term in Eq. 4, will not be identically zero as it was in the above analysis. This effect of this additional term, which expresses the convection of perturbation temperature, is destablizing, other things being equal, if $\langle \theta D \theta \rangle$ is positive, that is if $[\theta(1)]^2 > [\theta(0)]^2$. This last condition certainly holds if the lower boundary is conducting and the upper boundary is not conducting (and therefore less restrictive in the eigenvalue problem). Further, there will generally be an interaction between the two effects of throughflow. It is suggested that even in the general case the effect of a small amount of throughflow will be destablizing if that throughflow is in the direction from a more restrictive boundary to a less restrictive boundary.

Notation

a = wave number
D = differential operator, d/dz
F = unperturbed temperature gradient
P = modified thermal Peclet number
R = Rayleigh-Darcy number
R_c = critical value of R
z = vertical coordinate
Γ = mass flux perturbation
θ = temperature perturbation

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